OBSERVATIONS ON THE HYPERBOLA

$y^2 = 10x^2 + 6$

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Abstract

The binary quadratic equation $y^2 = 10x^2 + 6$ representing hyperbola is considered and a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a few special Pythagorean triangle each of which satisfying certain relations among its sides, area and perimeter, are obtained.

Keywords: binary quadratic, hyperbola, integral points, Pell equation.

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Introduction

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1, 2, 3, 4]. In [6] infinitely many Pythagorean triangles in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $y^2 = 3x^2 + 1$. In [7], a special Pythagorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$. In [13], different patterns of infinitely many Pythagorean triangles are obtained by employing the non-integral solutions of $y^2 = 12x^2 + 1$. In this context one may also refer [8 – 16]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 10x^2 + 6$ representing a hyperbola and the corresponding properties.

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Notations Used:

$t_{m,n}$ = Polygonal number of rank n with size m.

P_n^m = Pyramidal number of rank n with size m.

CP_n^m = Centered pyramidal number or rank n with size m.

$CP_{m,n}$ = Centered polygonal number or rank n with size m.

GNO_n = Gnomonic number of rank n.

S_n = Star number of rank n.

Method of Analysis:

The binary quadratic equation representing hyperbola under consideration is

$$y^2 = 10x^2 + 6 \qquad \textbf{(1)}$$

whose smallest positive integer solution is $x_0 = 1, y_0 = 4$.

To obtain the other solutions of (1), consider the Pellian equation $y^2 = 10x^2 + 1$ whose general solution $(\overline{x_n}, \overline{y_n})$ is given by

$$\overline{x_n} = \frac{g}{2\sqrt{10}} \text{ and } \overline{y_n} = \frac{f}{2} \text{ in which}$$

$f = (19 + 6\sqrt{10})^{n+1} + (19 - 6\sqrt{10})^{n+1}$ and $g = (19 + 6\sqrt{10})^{n+1} - (19 - 6\sqrt{10})^{n+1}$, $n = -1,0,1,2,3.....$

Applying Brahmagupta lemma between the solutions of (x_0, y_0) and $(\overline{x_n}, \overline{y_n})$ the general solution of (1) is found to be

$$x_{n+1} = \frac{f}{2} \pm \frac{2g}{\sqrt{10}} \qquad \textbf{(2)}$$

$$y_{n+1} = 2f \pm \frac{5g}{\sqrt{10}} \qquad \textbf{(3)}$$

where $n = -1,0,1,2,3....$

Thus ,(2) and (3) represent non-zero distinct integral solutions of hyperbola (1).

The recurrence relations satisfied by x and y are given by

$$x_{n+3} = 38x_{n+2} - x_{n+1}, \quad x_0 = 1, x_1 = 43$$

$$y_{n+3} = 38y_{n+2} - y_{n+1}, \quad y_0 = 4, y_1 = 136$$

Taking the positive signs in (2) and (3), some numerical examples of x and y satisfying (1) are given in the following table:

n	x_n	y_n
0	1	4
1	43	136
2	1633	5164
3	62011	196096
4	2354785	7446484
5	89429819	282770296

Taking the negative signs in (2) and (3), some numerical examples are represented below:

n	x_n	y_n
0	1	4
1	-5	16
2	-191	604
3	-7253	22936
4	-275423	870964
5	-10458821	33073696

From the above table, it is seen that the values of x_n are odd whereas that of y_n are even and $\equiv 0(\text{mod}\,4)$

Also, a few interesting properties between the solutions and special numbers are given below:

1. $S_f = 4[(2y_{2n+2} - 5x_{2n+2}) - (2y_{n+1} - 5x_{n+1})] + 13$

2. $3GNO_f = 4(2y_{n+1} - 5x_{n+1}) - 3$

3. $2t_{m,f} = \frac{2}{3}\{(m-2)[2y_{2n+2} - 5x_{2n+2}] - (m-4)[2y_{n+1} - 5x_{n+1}]\} + 2(m-2), m \geq 3$

4. $6P_f^m = \frac{2}{3}\{(m-2)(2y_{3n+3} - 5x_{3n+3}) + 3(2y_{2n+2} - 5x_{2n+2}) +$

$(2m-1)(2y_{n+1} - 5x_{n+1} + 1)\} + 6, m \geq 3$

5. $6CP_f^m = \frac{2}{3}\{m(2y_{3n+3} - 5x_{3n+3}) + 2(m+3)(2y_{n+1} - 5x_{n+1}), m \geq 3$

6. $2CP_{m,f} = \frac{2}{3}\{m[(m-2)(2y_{2n+2} - 5x_{2n+2}) + m(4-m)(2y_{n+1} - 5x_{n+1})] + 2m(m-2) + 2, m \geq 3$

7. $\frac{2}{3}[2y_{2n+2} - 5x_{2n+2}] + 2$ is a Perfect square.

8. $\frac{2}{3}[2y_{3n+3} - 5x_{3n+3}\} + 2[2y_{n+1} - 5x_{n+1}]$ is a Cubic integer.

9. $4[2y_{2n+2} - 5x_{2n+2}] + 12$ is a Nasty number.

Remarkable Observations:

1. Let N be any non-zero positive integer write $N = \frac{x_{n+1} - 1}{2}$.

   Treating N as the rank of the triangular number $t_{3,N}$ it is observed that $80t_{3,N} + 16$ is a Perfect square.

2. Let m and n be any two non-zero distinct positive integers such that $m = x_{s+1} + y_{s+1}, and. n = x_{s+1}$ Treat m and n as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$ where $\alpha = 2mn, \beta = m^2 - n^2, \gamma = m^2 + n^2, m > n > 0$. Then the Pythagorean triangle T satisfies the relations

   (i) $5\beta - 4\gamma - \alpha = 6$

(ii) $\gamma - \frac{4A}{P} = 5(\gamma - \beta) + 6$

(iii) $20\frac{A}{P} + \gamma = 6(\alpha + 1)$

(iv) $(\gamma - \alpha) = 10t_{4,x} + 6$

where A and P represent the Area and Perimeter of the Pythagorean Triangle T.

3. Employing the solutions (x, y) of (1), following relations among the special polygonal and Pyramidal numbers are obtained:

(i) $\left[\frac{3P_{y-2}^{3}}{t_{3,y-2}}\right]^2 = 10\left[\frac{P_x^{5}}{t_{3,x}}\right]^2 + 6$

(ii) $\left[\frac{3P_{y-2}^{3}}{t_{3,y-2}}\right]^2 = 10\left[\frac{2P_{x-1}^{5}}{t_{4,x-1}}\right]^2 + 6$

Conclusion:

In this paper, we have presented non-zero distinct integer solutions of the hyperbola $y^2 = 10x^2 + 6$ and employing these solutions, a special Pythagorean triangle has been obtained. As the binary quadratic equations are rich in variety due to their definition, one may search for other choices of hyperbola for their corresponding pattern of solutions along with the properties involving special numbers.

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